A One-Vendor Multiple-Buyer Production-Distribution System: The Value of Vendor Managed Inventory

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The paper considers a production/distribution system that involves one vendor and multiple buyers. The buyers belong to an entity that is operationally more cost efficient than the vendor. We compare vendor-managed inventory (VMI) with integrated production and inventory decision-making to the traditional decentralized setting in which the vendor and the buyers make independent decisions. Specifically for the two-buyer problem, we characterize the optimal solution under VMI which is more cost efficient than literature. For the general multiple-buyer problem, we develop a heuristic to approach decisions under both settings. One surprising finding is that under VMI, the production decision does not involve all operational information of the vendor or the buyers. Numerical results suggest that VMI and just-in-time production are substitutes in improving the efficiency of the distribution network. VMI can be more critical for companies with rigid capacity, seasonally low demand, or complex distribution networks.

Key words: production; inventory; vendor managed inventory; supply chain; game theory

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1. Introduction

In recent decades, vendor-managed inventory (VMI), whereby manufacturers/vendors store unsold items at the buyers' locations and take the responsibility of monitoring inventory levels and replenishing stocks, received great attention in both practice and academia. While there is a vast body of literature concerning VMI from many different aspects (Marquès et al. (2010)), our paper is most relevant to those modeling operational decisions in supply chains.

This stream of literature begins with the study of a one-vendor one-buyer supply chain. Early papers focused on joint economic lot size. Goyal (1977) and Banerjee (1986) are among the firsts that examine an integrated production-inventory system with infinite and finite production rate respectively. Improved production/inventory schedules are proposed in many subsequent works such as Goyal (1988), Hill (1997) and Hill (1999). Later studies take into account other realistic factors such quantity discount (Lal and Staelin (1984), Rosenblatt and Lee (1985), Munson and Rosenblatt (2001)), stochastic demand and lead time (Ben-Daya and Hariga (2004)), capacity constraint (Goyal (2000)), shortage (Zhou and Wang (2007)), pricing (Boyaci and Gallego (2002)), negative echelon holding cost (Hill and Omar (2006)).

The value and implementation of VMI, however, were not highlighted until recently. Particularly for the one-vendor one-buyer system, Dong and Xu (2002) analyze the channel benefit of VMI; Braglia and Zavanella (2003) consider implementing VMI through consignment stocks and Valentini and Zavanella (2003) provide more managerial insights on such contracts through case studies. Gümüs et al. (2008) compare the benefit of VMI where the vendor makes all ordering decisions, versus consignment inventory where the buyer places orders but the vendor bears a portion of the inventory holding costs. Nagarajan and Rajagopalan (2008) investigate the value of a holding-cost-subsidy contract in VMI where the vendor makes ordering decisions, as opposed to retailer-managed inventory where the buyer makes ordering decisions.

In coping with today's complex supply chain, it is paramount to understand extended systems comprising multiple buyers. Compared with the single-vendor-single buyer system, the one-vendor multi-buyer problems have been known non-trivial. Banerjee and Burton (1994) show that the optimal policy in a multi-buyer scenario can be quite different than that of a single buyer. Lu (1995) considers a one-vendor-multi-buyer setting in which each buyer procures a distinct type of product, and provides heuristics to solve the integrated inventory problem. Zavanella and Zanoni (2009) study a one-vendor-multi-buyer system with one common product, aiming to derive the optimal solution for the integrated system as well as the equilibrium solution for the decentralized system where the vendor and the buyers shoulder their own inventory costs. It has been noted that this analysis only applies to consecutive delivery patterns (Zavanella and Zanoni (2010)) among multiple buyers, which is shown less superior than rotational delivery patterns in Ben-Daya et al. (2013). Further, Goyal et al. (2012) note that the general one-vendor-multi-buyer problem (particularly the characterization of the total cost function) has not been fully explored in Zavanella and Zanoni (2009) or Ben-Daya et al. (2013), thus deserving more in-depth study. We are therefore motivated to continue this line of research by studying one-vendor-multi-buyer problems under a similar framework of Zavanella and Zanoni (2009) with rotational delivery patterns.

In this paper, we consider a vendor selling to multiple heterogeneous buyers. The buyers are owned by one entity, thus representing the distribution system of a large enterprise, e.g., Walmart, Target, Home Depot and many auto dealers. The vendor is a specialist (e.g., a just-in-time manufacturer) who incurs higher inventory holding cost than any of the buyers. It is then more expensive to keep the inventory at the vendor than at the buyers' site, which gives incentive for the vendor to shift the inventories downstream. In practice, this could be realized via various marketing and operational mechanisms including quantity discount, short term promotion, and particularly in this paper, vendor-managed inventory (VMI). From this aspect, unlike many papers (e.g., Lu (1995), Hill (1997)) which assume that the buyers' holding costs are higher than the vendor/manufacturer's (e.g., wholesalers are usually more cost efficient than retailers), the scenario of this problem is closer to those considered in Valentini and Zavanella (2003), Hill and Omar (2006), Nagarajan and Rajagopalan (2008), Zavanella and Zanoni (2009), etc.

We next describe the model in §2 and derive the total cost expression for the general multi-buyer problem in §3. Then we characterize solutions for both VMI and the decentralized system when there are two buyers (§4). Finally in §5, we develop heuristic solutions for the general multi-buyer problem. Managerial insights are summarized in §6.

2. Model

Consider $Y+1$ firms — a vendor (production rate P) taking orders from Y buyers (demand rate d_i for buyer i) on a continuous-time horizon. The production and demand rates are all deterministic.

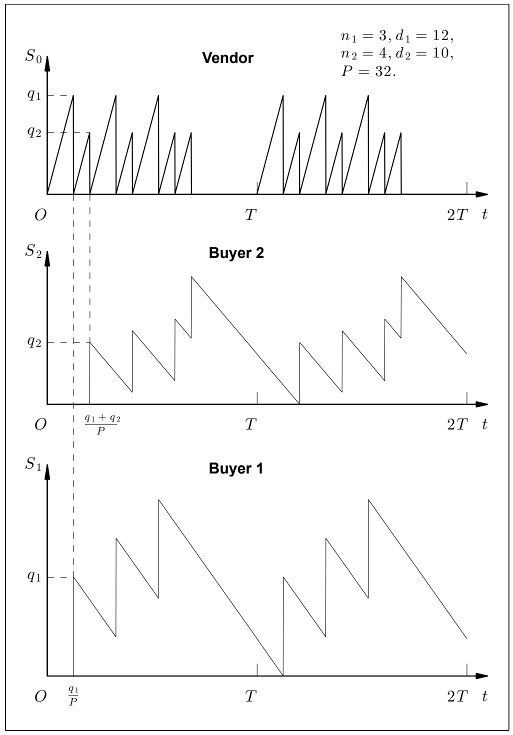

Figure 1 Inventory levels of the vendor (S_0) and two buyers (S_1, S_2): $P = 32, d_1 = 12, d_2 = 10, n_1 = 3, n_2 = 4$

We index $i = 0$ for the vendor and $i = 1, 2, ..., Y$ for the buyers. Let T denote the production cycle and n_i the number of deliveries to buyer i per cycle. In aligning with relevant literature, we assume that that the system requires there be at least one order per cycle, i.e., $n_i \geq 1$ for $i = 1, 2, ..., Y$.

Orders will be produced and delivered to each buyer at equal-size shipments. In particular, during each cycle the batches of the buyers will be delivered on a rotational basis (e.g., buyer 1, buyer 2, buyer 3, buyer 1, buyer 2, buyer 3, ...). If the delivery schedule for a buyer is finished for the cycle, then simply skip it and serve the next unfulfilled buyer on the list. The rotational delivery pattern initially appeared in Zavanella and Zanoni (2009), and Ben-Daya et al. (2013) show its superiority to consecutive delivery pattern (e.g., buyer 1, buyer 1, buyer 1, buyer 2, buyer 2, buyer 2, buyer 3, ... as in Zavanella and Zanoni (2010)). Figure 1 illustrates how this works in a one-vendor-two-buyer system.

The problem then calls for the following decisions to be made (the sequence and decision makers of these variables will be discussed later):

- Production cycle T for the vendor;
- Delivery frequency $\mathbf{n} = (n_1, n_2, ..., n_Y)$ for the buyers;
- Delivery pattern $\rho \in \mathcal{P} : \{[1], [2], ..., [Y]\} \to \{1, 2, ..., Y\}$ where $\rho([i]) \neq \rho([j])$ if $i \neq j$.

ρ describes the rotational delivery sequence among the Y buyers. The set $\{[1], [2], ..., [Y]\}$ is used to denote the original index of the buyers. Thus the rotation list always starts with buyer 1 and ends on Y, or equivalently, buyer $[\rho^{-1}(1)]$ and $[\rho^{-1}(Y)]$ respectively.

Denote A_i and h_i the setup cost and holding cost per unit-time per item for firm i. As discussed before, we only consider the case when $h_0 \geq h_i$ for $i = 1, 2, ..., Y$. Let TC_i represent the total cost in *a cycle* incurred at firm i's site, and $TC_{ave,i} = TC_i / T$ the average total cost at firm i's site.

We compare two kinds of systems: (1) **Vendor-Managed Inventory (VMI)** in which inventories are under the ownership of the vendor regardless of the storage site, and (2) a **decentralized system** in which the vendor and the buyers make independent decisions on production and inventories respectively.

Specifically under VMI, the vendor bears all the costs at each site, and will determine the production cycle T, delivery frequency for each buyer n_i and the delivery pattern ρ that minimize the total average cost of the system $TC_{ave,s} = \sum_{i=0}^{Y} TC_{ave,i}$. The optimal decisions are denoted as $(T^{**}, \mathbf{n}^{**}, \rho^{**})$. In the decentralized systems, assume that all buyers belong to the same entity. Given the production cycle T, the buyers will agree on a delivery frequency $(n_1, ..., n_Y)$ and pattern ρ that minimizes total cost of the buyers, $TC_{ave,b} = \sum_{i=1}^{Y} TC_{ave,i}$. On the other hand, upon any decision from the buyers' end, the vendor will focus on determining production cycle T that minimizes $TC_{ave,0}$. The equilibrium is denoted by $(T^*, \mathbf{n}^*, \rho^*)$.

Table 1 summarizes all notations used in this paper.

3. Total Cost for the Vendor and the Buyers

In this section, we derive the total cost expression at each firm's site, which has not been explicitly characterized in previous literature. We start from a simple one-vendor one-buyer problem ($Y = 1$). Analysis and notations can then naturally be extended to the multiple-buyer case.

P	Production rate for the vendor
d_j	Demand rate for buyer j, $j=1,2,..,Y$
A_0	Setup cost for the vendor
A_j	Setup cost for buyer j, $j=1,2,..,Y$
h_0	Holding cost for the vendor
h_j	Holding cost for buyer j, $j=1,2,..,Y$
T	Production cycle for the vendor
n_j	Number of orders that buyer j places in a cycle, $j=1,2,..,Y$
TC_0	Total cost for the vendor in a cycle
TC_j	Total cost for buyer j in a cycle, $j=1,2,..,Y$
$TC_b=\sum_{j=1}^{Y} TC_j$	Total cost for the buyers in a cycle
$TC_{sys} \sum_{j=0}^{Y} TC_j$	Total cost for the entire system in a cycle, $j=1,2,..,Y$
$TC_{ave,0}=TC_0/T$	Average total cost for the vendor
$TC_{ave,j}=TC_j/T$	Average otal cost for buyer j, $j=1,2,..,Y$
$TC_{ave,b}=\sum_{j=1}^{Y} TC_j/T$	Average total cost for the buyers
$TC_{ave,s} \sum_{j=0}^{Y} TC_j/T$	Average total cost for the entire system, $j=1,2,..,Y$

Table 1 Notations

3.1. Total cost in a one-vendor one-buyer problem

In this case, the vendor delivers n_1 batches at equal size $q_1=d_1T/n_1$ to the solo buyer (indexed 1) during each cycle. The vendor manufactures at rate P and ships a batch to the buyer once it is done. The *no-stockout condition* requires that $P \geq d_1$. Figure 2 illustrates the inventory levels of the vendor and the buyer in this setting. It can be observed that

- The k^{th} batch arrives at

$$a_k = k\frac{q_1}{P}$$

- The k^{th} batch is completely sold (and the $(k+1)^{st}$ batch can start to sell) by

$$t_k = a_1 + k\frac{q_1}{d_1} = \frac{q_1}{P} + k\frac{q_1}{d_1}$$

It can be observed that batches are generally delivered before the last one is completely sold, and the inventory of the buyer "piles up" during the course of the cycle. For example, the first batch is not completely gone until t_1, yet the second batch is delivered at $a_2 < t_1$. Thus the second batch is idle and untouched during $[a_2, t_1]$. Also the amount of idle time is extended for later batches (i.e., $t_1 - a_2 < t_2 - a_3 < t_3 - a_4$). Therefore for *any batch*, the inventory holding cost consists of two parts:

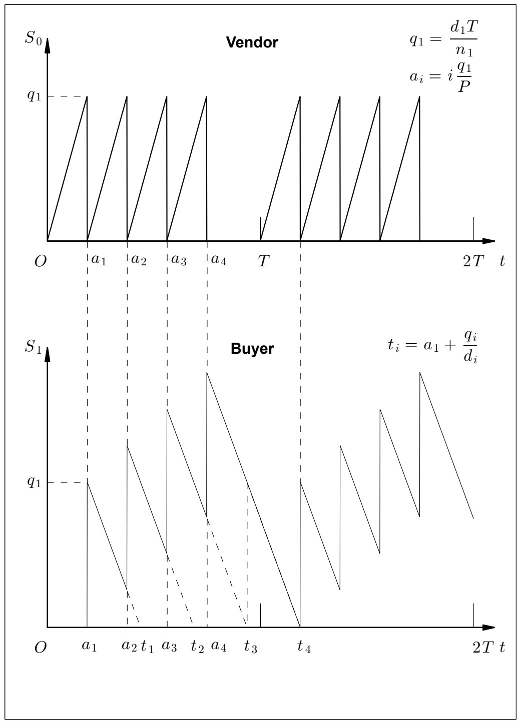

Figure 2 Inventory level with one vendor and one buyer

1. *Holding Cost due to Economies of Scale*: $\frac{q_1}{2}(t_k - t_{k-1})h_1$

2. *Holding Cost due to Idle Batch*: $h_1 q_1 W_{1,k}$ where $k > 1$ and $W_{1,k} = t_{k-1} - a_k = (k-1)(\frac{q_1}{d_1} - \frac{q_1}{P})$ is the idle time for buyer 1's k^{th} batch.

The total cost for the buyer in one cycle is then the sum of the set-up cost as well as two kinds of holding costs:

$$\begin{aligned} TC_1 &= A_1 n_1 + \sum_{k=1}^{n_1} \frac{q_1}{2}(t_k - t_{k-1})h_1 + \sum_{k=2}^{n_1} h_1 q_1 W_{1,k} = A_1 n_1 + h_1 \frac{q_1}{2} T + h_1 q_1 W_1 \\ &= A_1 n_1 + \frac{C_1}{n_1} + \frac{2C_1 W_1}{n_1 T} \end{aligned} \tag{1}$$

where $C_1 = \frac{h_1 d_1 T^2}{2}$ and $W_1 = \sum_{k=2}^{n_1} W_{1,k} = \sum_{k=2}^{n_1}(k-1)\left(\frac{q_1}{d_1} - \frac{q_1}{P}\right) = \frac{n_1(n_1-1)}{2}\frac{T}{n_1}\left(1 - \frac{d_1}{P}\right)$ is the cumulative batch idle time.

The total cycle cost for the vendor can be readily derived as

$$TC_0 = A_0 + h_0 \frac{q_1}{2}\frac{d_1 T}{P} = A_0 + \frac{h_0 d_1^2 T^2}{2n_1 P} \tag{2}$$

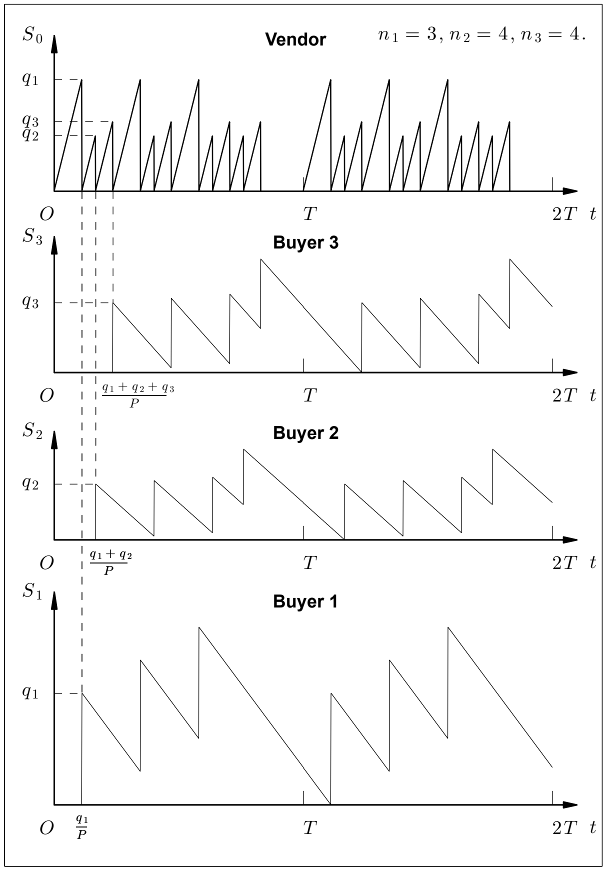

Figure 3 Inventory levels with one vendor and three buyers

3.2. Total cost in the general one-vendor multi-buyer problem

We now extend our analysis to the general Y-buyer case. Figure 3 illustrates the change of inventory levels where there are three buyers.

In general, the equal-batch size for buyer j is

$$q_j = d_j T/n_j, \quad for \quad j = 1, 2, ..., Y \tag{3}$$

The *no-stockout condition* in this case is a little bit different than that with one buyer, in which we only ask for the supply to cover demand, i.e., $P \geq d_1$. For multiple buyers, the vendor produces and delivers batches of each buyer on a rotational basis. For any buyer j, after getting its k^{th} order, it has to wait another round of production (that produces the a batch for buyer $j+1, ..., Y, 1, 2, ...j-1$, and j itself) before the $(k+1)^{th}$ order is delivered. Therefore, the batch q_j should be able to cover buyer j's demand during this period of time ($\sum_{i=1}^{Y} q_i/P$). That is

$$\sum_{i=1}^{Y} q_i/P \le q_j/d_j \quad \forall j = 1, 2, ..., Y$$

By (3) the *no-stockout condition* is practically

$$P \ge \max_{j=1,2,...,Y}\{n_j\} \sum_{i=1}^{Y} \frac{d_i}{n_i}. \tag{4}$$

Specifically when $Y = 1$, the condition is degenerated to $P \ge d_1$. We therefore assume (4) for the general problem.

Recall that batches of the buyers are delivered on a rotational mode in each cycle, and fulfilled buyers will be skipped. We can therefore derive the following for buyer j:

- The k^{th} batch arrives at

$$a_{j,k} = \sum_{l=1}^{j} \frac{q_l}{P} \min\{n_l, k\} + \sum_{j+1}^{Y} \frac{q_l}{P} \min\{n_l, k-1\}$$

- The k^{th} batch gets completely sold (and the $(k+1)^{th}$ batch can start selling) at

$$t_{j,k} = a_{j,1} + k\frac{q_j}{d_j} = \sum_{l=1}^{j} \frac{q_l}{P} + k\frac{q_j}{d_j}.$$

Similar to that of the one-buyer case, the total cost for buyer j's k^{th} batch consists of two parts:

1. *Holding Cost due to Economies of Scale*: $\frac{q_j}{2}(t_{j,k} - t_{j,k-1})h_j$
2. *Holding Cost due to Idle Batch*: $h_j q_j W_{j,k}$ where $k > 1$ and

$$W_{j,k} = t_{j,k-1} - a_{j,k} = \sum_{l=1}^{j} \frac{q_l}{P} + (k-1)\frac{q_j}{d_j} - \sum_{l=1}^{j} \frac{q_l}{P} \min\{n_l, k\} - \sum_{j+1}^{Y} \frac{q_l}{P} \min\{n_l, k-1\} \tag{5}$$

is the idle time for buyer j's k^{th} batch.

The total cost for buyer j in a cycle is then

$$\begin{aligned} TC_j &= A_j n_j + \sum_{k=1}^{n_j} \frac{q_j}{2}(t_k - t_{k-1})h_j + \sum_{k=2}^{n_j} h_j q_j W_{j,k} = A_j n_j + h_j \frac{q_j}{2} T + h_j q_j W_j \\ &= A_j n_j + \frac{C_j}{n_j} + \frac{2C_j}{T}\frac{W_j}{n_j} \end{aligned} \tag{6}$$

where $C_j = h_j d_j T^2/2$ and $W_j = \sum_{k=2}^{n_j} W_{j,k}$ is the cumulative batch idle time for buyer j.

The total cycle cost for the vendor is

$$TC_0 = A_0 + h_0 \sum_{j=1}^{Y} \frac{q_j}{2}\frac{d_j T}{P} = A_0 + \frac{h_0 T^2}{2P} \sum_{j=1}^{Y} \frac{d_j^2}{n_j} \tag{7}$$

4. The Two-Buyer Problem

We now consider the problem with exactly two buyers, i.e., $Y=2$. The goal is to derive optimal solution under VMI and the equilibrium for RMI.

By (7), the vendor's average cost is

$$TC_{ave,0}=\frac{TC_0}{T}=\frac{A_0}{T}+\frac{h_0T}{2P}\left(\frac{d_1^2}{n_1}+\frac{d_2^2}{n_2}\right)$$

The buyers' cost when $Y=2$ are given in Goyal et al. (2012). We provide a brief proof in the appendix, and conduct optimum/equilibrium analysis based on these cost functions.

LEMMA 1. **Total average cost for the buyers when $Y=2$.** *(Goyal, Huang and Li 2012)*

$$\begin{aligned}&TC_{ave,1}\\&=\begin{cases}\dfrac{A_1n_1}{T}+\dfrac{h_1d_1T}{2}-\dfrac{h_1d_1T}{2P}(n_1-1)(\dfrac{d_1}{n_1}+\dfrac{d_2}{n_2}) & if\ \ n_1\le n_2\\ \dfrac{A_1n_1}{T}+\dfrac{h_1d_1T}{2}-\dfrac{h_1d_1T}{2n_1P}\left[(d_1+d_2)(n_1-1)-d_2(n_2-n_1)\right] & if\ \ n_1>n_2\end{cases}\end{aligned}$$

$$\begin{aligned}&TC_{ave,2}\\&=\begin{cases}\dfrac{A_2n_2}{T}+\dfrac{h_2d_2T}{2}-\dfrac{h_2d_2T}{2n_2P}\left[(\dfrac{d_1}{n_1}+\dfrac{d_2}{n_2})n_2(n_2-1)\right.\\ \qquad\qquad\qquad\qquad\left.-\dfrac{d_1}{n_1}(n_2-n_1)(n_2-n_1+1)\right] & if\ \ n_1\le n_2\\ \dfrac{A_2n_2}{T}+\dfrac{h_2d_2T}{2}-\dfrac{h_2d_2T}{2P}(n_2-1)(\dfrac{d_1}{n_1}+\dfrac{d_2}{n_2}) & if\ \ n_1>n_2\end{cases}\end{aligned}$$

In what follows, we first characterize equilibrium solutions in the decentralized system, and then the integrated optimal solution for VMI.

4.1. Equilibrium in the decentralized system

In the decentralized system, the vendor chooses the optimal cycle T that minimizes its cost $TC_{ave,0}$, and buyers determine order frequencies (n_1,n_2) and pattern ρ that minimize the total cost $TC_{ave,b}=TC_{ave,1}+TC_{ave,2}$. We aim at solving the equilibrium for this system, which is similar to the "sequential solution" in Zavanella and Zanoni (2009).

First of all, given the buyers' decisions, it is not hard to verify that the vendor will set the production cycle as

$$T^* = \sqrt{\frac{2A_0P}{h_0\left(\frac{d_1^2}{n_1}+\frac{d_2^2}{n_2}\right)}} \tag{8}$$

On the other than, the buyers will configure an optimal order frequency (n_1, n_2) and delivery pattern ρ based upon the vendor's production cycle T. To solve this problem, first assume that the delivery pattern ρ is chosen. The first-order condition (FOC) of buyers' cost follows immediately after Lemma 1:

$$\frac{\partial}{\partial n_1}TC_{ave,b}|_\rho =$$
$$\begin{cases} \frac{A_1}{T} - \frac{T}{2P}\left[\frac{d_1^2h_1}{n_1^2} + \frac{d_1d_2(h_1-h_2)}{n_2} + \frac{2d_1d_2h_2}{n_1^2}\right] & if\ \ n_1 \le n_2 \\ \frac{A_1}{T} - \frac{T}{2P}\left[\frac{d_1^2h_1 + d_1d_2(h_1+h_2+h_1n_2-h_2n_2)}{n_1^2}\right] & if\ \ n_1 > n_2 \end{cases}$$

$$\frac{\partial}{\partial n_2}TC_{ave,b}|_\rho =$$
$$\begin{cases} \frac{A_2}{T} - \frac{T}{2P}\left[\frac{d_2^2h_2 - d_1d_2(h_1-h_2)(n_1-1)}{n_2^2}\right] & if\ \ n_1 \le n_2 \\ \frac{A_2}{T} - \frac{T}{2P}\left[\frac{d_2^2h_2}{n_2^2} + \frac{d_1d_2(h_2-h_1)}{n_1}\right] & if\ \ n_1 > n_2 \end{cases}$$

Solving $\min_{n_1,n_2} TC_{ave,b}$ would involve two sub-problems for the regions of $n_1 \le n_2$ and $n_1 > n_2$, and they may not be convex for general parameters. However, in practice buyers ordering the same kind of products from the vendor are highly likely to incur the same storage spaces, electricity usages, insurances, etc., which yield the same holding costs. Therefore we focus on examining the scenario when $h_1 = h_2 = h$. In this case, the FOCs are identical for both regions ($n_1 \ge\le n_2$):

$$\frac{\partial}{\partial n_1}TC_{ave,b}|_\rho = \frac{A_1}{T} - \frac{T}{2P}\frac{d_1(d_1+2d_2)h}{n_1^2}$$
$$\frac{\partial}{\partial n_2}TC_{ave,b}|_\rho = \frac{A_2}{T} - \frac{T}{2P}\frac{d_2^2h}{n_2^2}$$

We can therefore solve the buyers' order frequencies as

$$\begin{aligned} n_1^*(\rho) &= \sqrt{\frac{d_1(d_1+2d_2)hT^2}{2A_1P}} \\ n_2^*(\rho) &= \sqrt{\frac{d_2^2hT^2}{2A_2P}} \end{aligned} \tag{9}$$

and the minimum buyers' cost is

$$TC^*_{ave,b}(\rho) = \frac{hT}{2P}(d_1+d_2)(P-d_1-d_2) + \sqrt{\frac{2h}{P}}(\sqrt{A_1}\sqrt{d_1(d_1+2d_2)} + \sqrt{A_2}d_2).$$

Buyers' delivery pattern can be solved by optimizing the above total cost, i.e., $\rho^* = \arg\min_{\rho\in\mathcal{P}} TC^*_{ave,b}(\rho)$. The solution is given by:

$$\begin{aligned} &\rho^*([i]) \\ &= \begin{cases} i & if \ \ \sqrt{A_{[1]}}\sqrt{d_{[1]}(d_{[1]}+2d_{[2]})} + \sqrt{A_{[2]}}d_{[2]} \le \sqrt{A_{[2]}}\sqrt{d_{[2]}(d_{[2]}+2d_{[1]})} + \sqrt{A_{[1]}}d_{[1]} \\ j & o/w \end{cases} \end{aligned} \tag{10}$$

where $i, j \in \{1,2\}$ and $i \ne j$.

Solving (8), (9) and (10) together, we derive the following result regarding equilibrium decisions between the vendor and the buyers:

Proposition 1. **(Equilibrium in a one-vendor-two-buyer system)** *In a decentralized system with two buyers of identical holding cost, i.e., $h_1 = h_2 = h$, the equilibrium production cycle and order frequencies are:*

$$\begin{aligned} T^* &= \frac{\sqrt{2Ph}}{h_0}\frac{A_0}{\sqrt{\frac{d_1}{d_1+2d_2}}d_1\sqrt{A_1} + d_2\sqrt{A_2}} \\ n_1^* &= \frac{h}{h_0}\frac{\sqrt{d_1(d_1+2d_2)}A_0}{\sqrt{\frac{d_1}{d_1+2d_2}}d_1A_1 + d_2\sqrt{A_1A_2}} \\ n_2^* &= \frac{h}{h_0}\frac{d_2A_0}{\sqrt{\frac{d_1}{d_1+2d_2}}d_1\sqrt{A_1A_2} + d_2A_2}, \end{aligned} \tag{11}$$

and the delivery pattern ρ^* follows (10).

Specifically, one may observe from (10) that the optimal delivery pattern matters with A_i and d_i for $i = 1,2$ only. It does not depend upon the production rate P or cost structure (A_0, h_0) of the vendor. This gives rise the following corollary:

Corollary 1. *For the decentralized system, the delivery pattern ρ is an internal decision that is not affect by upstream operational efficiencies.*

4.2. Optimal solution for VMI

For vendor-managed inventory, we enforce the same condition that $h_1 = h_2 = h$. The goal is to find $(T^{**}, n_1^{**}, n_2^{**})$ that minimize the system's total cost $TC_{ave,s} = \sum_{i=0}^{2} TC_{ave,i}$.

To solve this problem, first assume that the delivery pattern ρ is fixed. The FOCs of the total cost regarding production cycle and order frequencies can be derived as

$$\begin{aligned}
\frac{\partial}{\partial T} TC_{ave,s}|_\rho = & -\frac{A_0 + A_1 n_1 + A_2 n_2}{T^2} + \frac{h}{2P}\left[(\frac{1}{n_1} - 1)d_1^2 + (\frac{1}{n_2} - 1)d_2^2 + 2(\frac{1}{n_1} - 1)d_1 d_2 + (d_1 + d_2)P\right] \\
& + \frac{h_0}{2P}\left(\frac{d_1^2}{n_1} + \frac{d_2^2}{n_2}\right) \\
\frac{\partial}{\partial n_1} TC_{ave,s}|_\rho = & \frac{A_1}{T} - \frac{T}{P}\frac{d_1\left[d_1(h + h_0) + 2d_2 h\right]}{2n_1^2} \\
\frac{\partial}{\partial n_2} TC_{ave,s}|_\rho = & \frac{A_2}{T} - \frac{T}{P}\frac{d_2^2(h + h_0)}{2n_2^2}
\end{aligned}$$

through which we can identify the optimum:

$$\begin{aligned}
T^{**}(\rho) &= \sqrt{\frac{2A_0 P}{h(d_1 + d_2)(P - d_1 - d_2)}} \\
n_1^{**}(\rho) &= \sqrt{\frac{A_0 d_1[d_1(h + h_0) + 2d_2 h]}{A_1 h(d_1 + d_2)(P - d_1 - d_2)}} \\
n_2^{**}(\rho) &= \sqrt{\frac{A_0 d_2^2(h + h_0)}{A_2 h(d_1 + d_2)(P - d_1 - d_2)}}
\end{aligned} \tag{12}$$

and the minimal total cost of the system

$$T_{ave,sys}^{**}(\rho) = \sqrt{\frac{2}{P}}\left[\sqrt{A_0 h D(P - D)} + \sqrt{A_1 d_1(d_1(h_0 + h) + 2hd_2)} + \sqrt{A_2 d_2^2(h_0 + h)}\right].$$

We then solve the optimal delivery pattern via finding the one that minimizes $T_{ave,s}^{**}(\rho)$:

$$\rho^{**} = \arg\min_{\rho \in \mathcal{P}} T_{ave,sys}^{**}(\rho) = \arg\min_{\rho \in \mathcal{P}} \sqrt{A_1 d_1(d_1(h_0 + h) + 2hd_2)} + \sqrt{A_2 d_2^2(h_0 + h)} \tag{13}$$

The results on VMI are summarized as follows.

Proposition 2. **(Vendor-managed inventory in a one-vendor-two-buyer system)** *Under VMI with two buyers of identical holding cost, i.e., $h_1 = h_2 = h$, the production cycle T^{**} and order frequencies $\mathbf{n}^{**}$ that minimize the system's total cost is characterized by* (12) *and the delivery pattern ρ^{**} follows* (13).

Compared with Corollary 1, one can tell that under VMI, the delivery pattern will involve operational performance on both the vendor side (h_0) and the buyer side ($A_{i>0,}d_i, h$). However, the decision of production cycle only relies on market information (d_i) of the buyers. The set up costs of the buyers ($A_{i>0}$) and the holding cost of its own (h_0) would not affect the vendor's production schedule. These give rise to the following corollary.

COROLLARY 2. *Under VMI, the vendor only needs to know buyers' market condition and holding cost in determining its production schedule. The buyers set up cost or the vendor's holding cost would not affect the optimal production cycle.*

4.3. Numerical examples

We use the same parameters as in Goyal (1988), Zavanella and Zanoni (2009) and Ben-Daya et al. (2013):

$$P = 3200, d_{[1]} = 500, d_{[2]} = 1000, A_0 = 400, A_{[1]} = 75, A_{[2]} = 25, h_0 = 5, h = 4$$

to illustrate our results on VMI and its value compared to the decentralized system. Numerical solutions are summarized in the following table:

	$\rho([1],[2])$	n_1	n_2	T	$TC_{ave,0}$	$TC_{ave,1}$	$TC_{ave,2}$	$TC_{ave,b}$	$TC_{ave,sys}$
VMI	(2, 1)	5	2	0.501	925.6	781.8	729.9	1511.7	2437.4
Decentralized System	(2, 1)	12	2	1.627	510.7	1616.7	1550	3166.7	3677.4

It can be observed that our VMI solution $\rho([1],[2]) = (2,1)$, $\mathbf{n} = (5,2)$ and $T = 0.5$ yields an average total cost 2437.4 for the system — a nearly 51% saving than the decentralized system. Further, the average total cost is 6.1% lower than that in Zavanella and Zanoni (2009) (which suggests that the optimal solution be $\rho([1],[2]) = (1,2)$, $\mathbf{n} = (1,3)$ and $T = 0.425$ with total cost 2585.7 — verified by Goyal et al. (2012)), and 14.8% lower than in Ben-Daya et al. (2013) (which assumes common delivery frequency of all buyers, yielding optimal solution $\mathbf{n} = (2,2)$ and $T = 0.429$ with total cost 2797.9). This highlights the benefit of incorporating buyer delivery pattern into integrated production-inventory decision making, which, to our best knowledge, is uniquely considered in our paper among this line of research.

5. A heuristic for the General One-Vendor-Multiple-Buyer Problem

In the previous section, we have fully characterized solutions of the two-buyer problem in both VMI and the decentralized systems. As the number of buyers grows, the general total cost function (6) and (5) imply that the problem will become more much complex and even intractable. Nevertheless, the analysis in §4 does provide us with a good start point to tackle the general problem. We analyze problem with $Y > 2$ in the same spirit, and propose a heuristic for calculating the total cost. Equilibrium and optimal solutions are derived thereafter.

We first derive the total cost TC_j with given ρ assuming that $n_1 \le n_2 \le ... \le n_Y$. For buyer 1, by (5) and (6) the total cost is

$$TC_1 = A_1 n_1 + \frac{C_1}{n_1} + \frac{2C_1}{n_1}\frac{W_1}{T} = A_1 n_1 + C_1\left[1 - (n_1 - 1)\sum_{i=1}^{Y}\frac{d_i}{n_i P}\right].$$

For buyer 2,

$$TC_2 = A_2 n_2 + C_2\left[1 - (n_2 - 1)\sum_{i=1}^{Y}\frac{d_i}{n_i P} + \frac{(n_2 - n_1)(n_2 - n_1 + 1)}{n_2}\frac{d_1}{n_1 P}\right],$$

and similarly, the total cost for buyer 3 is

$$TC_3 = A_3 n_3 + C_3\left[1 - (n_3 - 1)\sum_{i=1}^{Y}\frac{d_i}{n_i P} + \sum_{i=1}^{2}\frac{(n_3 - n_i)(n_3 - n_i + 1)}{n_3}\frac{d_i}{n_i P}\right].$$

It can be verified that the same pattern holds for higher indexed buyers and

$$TC_j = A_j n_j + C_j\left[1 - (n_j - 1)\sum_{i=1}^{Y}\frac{d_i}{n_i P} + \sum_{i=1}^{j-1}\frac{(n_j - n_i)(n_j - n_i + 1)}{n_j}\frac{d_i}{n_i P}\right] \tag{14}$$

Note that the above cost functions are under the assumption that $n_1 \le n_2 \le ... \le n_Y$. In general, one should enforce all ranking assumptions on $(n_1, ..., n_Y)$, solve the sub-optimal total cost under each of them, and compare for global optimum. However, it has been shown in the two-buyer problem that if holding costs are the same across all buyers ($h_j = h$), the total cost for buyer j, TC_j, is irrelevant to the ranking assumption of $(n_1, ..., n_Y)$. We carry this insight to the general problem and define (14) as the *heuristic total cost* of buyer j in the one-vendor-multi-buyer system with identical holding cost ($h_1 = ... = h_Y = h$). The solutions for both the VMI and the decentralized system will be derived upon the heuristic total cost.

5.1. Equilibrium in the decentralized system

Consider that the buyers together determine the delivery pattern ρ and frequencies $(n_1,...,n_Y)$, and the vendor makes its own decision on the production cycle T. Solving the FOCs of the average total cost ($1/T$ of (7) and (14)) on given ρ, the optimal response functions for the vendor and the buyers are

$$T^*(\rho)=\sqrt{\frac{2A_0P}{h_0\sum_{i=1}^{Y}\frac{d_i^2}{n_i}}},\qquad n_j^*(\rho)=T\sqrt{\frac{d_j(d_j+2D_j)h}{2A_jP}},\quad j=1...Y. \tag{15}$$

where $D_j=\sum_{i=j+1}^{Y}d_i$ and $D=D_0$.

The optimal delivery pattern can be found through minimizing the buyers total cost $TC^*_{ave,b}(\rho)$. By (15), $TC^*_{ave,b}(\rho)=\frac{hT^*}{2P}D(P-D)+\sqrt{\frac{2h}{P}}\sum_{i=1}^{Y}\sqrt{A_id_i(d_i+2D_i)}$. Therefore,

$$\rho^*=\arg\min_{\rho\in\mathcal{P}}TC^*_{ave,b}(\rho)=\arg\min_{\rho\in\mathcal{P}}\sum_{i=1}^{Y}\sqrt{A_id_i(d_i+2D_i)} \tag{16}$$

The equilibrium can thus be derived as follows:

Proposition 3. **(Equilibrium in a general one-vendor-multi-buyer system)** *With heuristic total cost function* (14)*, consider a decentralized system with buyers of identical holding cost, i.e., $h_j=h$, $\forall j=1,2,...,Y$. The vendor and the buyers will make the following production-inventory decisions in equilibrium:*

$$\begin{aligned}T^* &= \frac{2A_0P}{h_0}\left(\sum_{i=1}^{Y}\frac{d_i^2}{B_i}\right)^{-1}\\ n_j^* &= \frac{2A_0P}{h_0}B_i\left(\sum_{i=1}^{Y}\frac{d_i^2}{B_i}\right)^{-1}\end{aligned}$$

where $B_j=\sqrt{\frac{d_j(d_j+2D_j)h}{2A_jP}}$, $D_j=\sum_{i=j+1}^{Y}d_i$, $D=D_0$, and the optimal delivery pattern ρ^* follows (16)*.*

5.2. Optimal solution under VMI

Now consider VMI. In minimizing the total cost of the system we have to make sure the following conditions hold under given delivery pattern ρ

$$\frac{\partial}{\partial T}TC_{ave,sys}|_{\rho} = -\frac{1}{T^2}(A_0+\sum_{i=1}^{Y}A_in_i)+\frac{h_0}{2P}\sum_{i=1}^{Y}\frac{d_i^2}{n_i}+\frac{h}{2P}\left[\sum_{i=1}^{Y}d_i(P-D)+\sum_{j=1}^{Y}\frac{d_j}{n_j}(d_j+2D_j)\right]=0$$

$$\frac{\partial}{\partial n_j}TC_{ave,sys}|_{\rho} = \frac{A_j}{T}-\frac{d_jT}{2n_j^2P}\left[(d_j+2D_j)h+d_jh_0\right]=0$$

where $D_j=\sum_{i=j+1}^{Y}d_i$ and $D=D_0$.

Solving the above we have the optimal solution:

$$T^{**}(\rho) = \sqrt{\frac{2PA_0}{hA_iD(P-D)}} \tag{17}$$

$$n_j^{**}(\rho) = \sqrt{\frac{A_0d_j\left[(h+h_0)d_j+2hD_j\right]}{A_jhD(P-D)}}$$

and

$$T^{**}_{ave,sys}(\rho)=\sqrt{\frac{2}{P}}\left[\sqrt{A_0hD(P-D)}+\sum_{i=1}^{Y}\sqrt{A_id_i(d_i(h_0+h)+2hD_i)}\right] \tag{18}$$

where $D_j=\sum_{i=j+1}^{Y}d_i$ and $D=D_0$.

The optimal delivery pattern is therefore

$$\rho^{**}=\arg\min_{\rho\in\mathcal{P}}TC^{**}_{ave,sys}(\rho)=\arg\min_{\rho\in\mathcal{P}}\sum_{i=1}^{Y}\sqrt{A_id_i(d_i(h_0+h)+2hD_i)} \tag{19}$$

The findings on optimal decisions under VMI are summarized in the following proposition.

PROPOSITION 4. **(Vendor-managed inventory in a general one-vendor-multi-buyer system)** *Under the heuristic total cost function* (14), *consider VMI with buyers of identical holding cost, i.e.,* $h_j=h$, $\forall j=1,2,...,Y$. *The optimal production cycle, delivery frequency and delivery pattern follow* (17) *and* (19) *respectively.*

Proposition 4 provides a general solution based on the heuristic total cost. Further, we can explicitly derive the optimal delivery pattern ρ^{**} when the buyers share identical cost structure ($A_i=A$, $h_i=h$ for $i=1,2,...,Y$) yet heterogeneous demand rates.

PROPOSITION 5. **(Optimal delivery pattern for identical buyers under VMI)** *For buyers with identical cost structure ($A_i=A, h_i=h$ for $i>0$), it is optimal to put buyers with higher demand on top of the rotation list. That is, assume that* $d_{[1]}\le d_{[2]}\le ...\le d_{[Y]}$, *then* $\rho^{**}([i])=Y+1-i$.

# of Buyers	Prod. Rate	Decentralized System			VMI			Eff. Loss
Y	P	**n**	T	$TC_{ave,sys}$	**n**	T	$TC_{ave,sys}$	δ
2	3200	(14, 8)	1.353	3034.1	$(7,5)$	0.116	2454.1	23.63%
4	6400	$(14,12,9,5)$	1.256	5266	(5, 4, 4, 3)	0.082	3854.7	36.61%
6	9600	$(14,13,11,$ $9,7,4)$	1.217	7450	(4, 4, 3, 3, 2, 2)	0.067	5108.1	45.84%
8	12800	$(14,13,12,11,$ $9,8,6,4)$	1.194	9613.3	(3, 3, 3, 3, 2, 2, 2, 1)	0.058	6287.5	52.9%

Table 2 Decentralized System vs. VMI as the number of buyers increases, $A_0 = 400, h_0 = 5,$ $d_i = 1200, A_i = 25, h_i = 4$

It practically suggests that in a distribution network where each location incurs similar operational cost (ordering and holding), the rotational delivery should start from fulfilling buyers with the highest demand to those with lowest demand. Therefore Proposition 4 and 5 together provides a handy VMI prescription for large distribution systems with identical buyers.

5.3. Numerical examples

We next conduct numerical experiments to illustrate how the outcomes are impacted by the number of buyers. Similar parameters as in §4.3 are adopted: $A_0 = 400, h_0 = 5$ and $d_i = 1200, A_i = 25, h_i = 4$ for $i = 1, 2, ..., Y$. To examine the impact of the number of buyers, we increase the production rate P in accordance with Y, such that the system's utilization rate remains the same as the system grows, i.e., $\sum_{i=1}^{Y} d_i/P = 75\%$. The results are summarized in Table 2. Specifically, the central columns are the equilibrium/optimal solution under the decentralized system (DS) and VMI. The last column shows the efficiency loss in DS compared to VMI. That is, $\delta = (TC^{DS}_{ave,sys} - TC^{VMI}_{ave,sys})/TC^{VMI}_{ave,sys}$.

It can be observed that as the number of buyers Y increases, the loss of efficiency also increases. This highlights the importance of vendor-managed inventory in large-scale distribution system. We also conduct similar analysis with different system utilities (Figure 4). It suggests that the efficiency gap can be significantly reduced if the system utilization is high. In other words, the

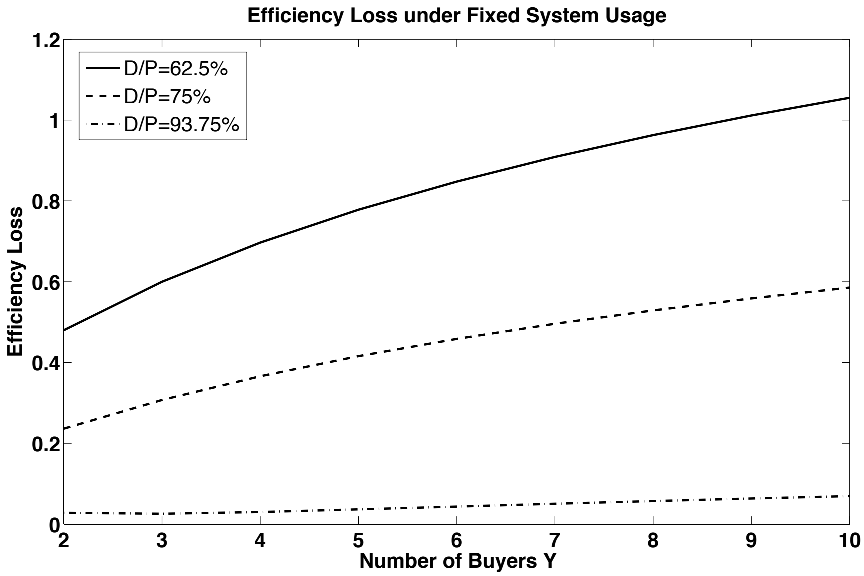

Figure 4 Efficiency Loss under fixed system utilization (62.5%, 75% and 93.75%)

impact of vendor-managed system is less obvious under flexible production which can be coped with the demand.

Thus far the numerical examples suggest two ways to enhance system performance. One is vendor-managed inventory (VMI), and the other is just-in-time (JIT) production. Most interestingly, these two measures are substitutes to each other and VMI can be more critical under rigid production system with high capacity, or when demand is low ($P >> D$).

6. Endnotes

This paper examines the application of vendor-managed inventory (VMI) with integrated production-inventory decision making compared to the traditional decentralized system in a one-vendor-multiple-buyer system. We develop general total cost function for the vendor and the buyers, which adapts to a much wider set of scenarios than existing literature. For distribution systems consisting of two buyers, we explicitly derive the optimal and equilibrium solutions (production, order frequency and delivery pattern) for both VMI and the decentralized system. For general distribution systems, we provide a heuristic to approach solutions when buyers share similar cost structures. The results demonstrate significant improvement than existing solutions in literature.

Production Cycle (T)	Buyers' Information *Operations*	*Market*
VMI	Holding cost only	✓
Decentralized System	✓	✓

Table 3 Information required for the decision of **production cycle** T

The model is particularly suitable for supply chains where a downstream firm operates many similar distribution centres and has greater logistics advantage than the upstream vendor. This include industries such as retailing (e.g., Walmart, Home Depot) and automobile (dealers order from a single manufacturer). The findings also generate several interesting implications for managerial decision making.

The first insight relates to the information requirements in reaching equilibrium/optimal outcomes (Table 3, 4, and 5 summarize the comparison between VMI and the decentralized system). Basically, there is less difference between the two settings in making delivery frequency decisions (Table 4) — vendor's operational information is a must under both settings. For the production cycle (Table 3), while the vendor will always need full information from the buyers in the decentralized system, less will be demanded under VMI. In particular, under VMI the vendor would only need substantially less operational information from the buyers — the set-up cost efficiency matters little in determining the production cycle. However, such is reversed when buyers determine delivery patterns (Table 5). Decisions on delivery pattern can be irrelevant to vendor's cost structure in the decentralized system, while this information is a must for VMI.

The second insight is on understanding the value of vendor-managed inventory. In general, VMI generates more savings as the distribution system involves more buyers. This impact is magnified when it is expensive to adjust production according to total demand rate. Such can happen in many industries such as automobile, where short-run capacity abandonment is unusual, or when an industry is highly unionized.

Delivery Frequency (**n**)	Vendor's Information *Operations*
VMI	✓
Decentralized System	✓

Table 4 Information required for the decision of **delivery frequency n**

Delivery Pattern (ρ)	Vendor's Information *Operations*
VMI	✓
Decentralized System	—

Table 5 Information required for the decision of **delivery pattern** ρ

Finally, we acknowledge that there are limits to this study, which, on the other hand, generates interesting future research opportunities. For example, while we managed to characterize the exact cost function for each stakeholder, in deriving the optimal and equilibrium solutions we frequently consider buyers with identical holding cost. Relaxing the assumption of identical holding cost would be an interesting extension to this line of research. In additional, on evaluating the benefit of VMI, we have been focusing on the total cost of the system. Another possibility is to analyze the impact on individual vendor/buyer, and see whether the VMI is offering sufficient incentivizes for individual stakeholders to participate. Followed by Figure 4, it would also be meaningful to explore whether it is beneficial to increase or decrease the number of the buyers to some extent, such that the total cost is to the best interest for the system.

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Appendix

Proof of Lemma 1 We show how to get the average total cost for buyer 1 when $n_1 \leq n_2$. Others can be proved in a similar fashion.

By (5), $W_{1,k} = (k-1)\left(\frac{q_1}{d_1} - \frac{q_1}{P} - \frac{q_2}{P}\right)$ hence

$$W_1 = \frac{n_1}{2}(n_1-1)\left(\frac{q_1}{d_1} - \frac{q_1}{P} - \frac{q_2}{P}\right) = \frac{n_1}{2}(n_1-1)\left(\frac{T}{n_1} - \frac{d_1 T}{n_1 P} - \frac{d_2 T}{n_2 P}\right)$$

Substitute this and (3) into (6), buyer 1's average cost is then

$$TC_{ave,1} = \frac{A_1 n_1}{T} + \frac{h_1 d_1 T}{2} - \frac{h_1 d_1 T}{2}(n_1-1)\left(\frac{d_1}{n_1 P} + \frac{d_2}{n_2 P}\right).$$

□

Proof of Proposition 5 We first propose the following lemma which will be used in the main proof.

Lemma A1. *Suppose $a, b, c, d \geq 0$ satisfying*

1. $a + b = c + d$, *and*
2. $|a - b| \leq |c - d|$,

then there is $\sqrt{a} + \sqrt{b} \geq \sqrt{c} + \sqrt{d}$.

Proof. It is sufficient to show that $(\sqrt{a} + \sqrt{b})^2 \geq (\sqrt{c} + \sqrt{d})^2$, which by the first condition, can be reduced to $ab \geq cd$.

By the second condition, there is $|a-b|^2 \leq |c-d|^2$, hence $(a+b)^2 - 4ab \leq (c+d)^2 - 4cd$. Applying the first condition, we can verify immediately that $ab \geq cd$. □

For $\lambda = \frac{h_0 + h}{2h} \geq 1$, it is sufficient to show that $\sum_{i=1}^{Y} \sqrt{d_i(d_i\lambda + D_i)}$ reaches the minimum value when $d_1 \geq \cdots \geq d_Y$. Suppose this is not true, then at the minimum there exists some $1 \leq t \leq Y-1$ such that $d_t < d_{t+1}$.

Define a new sequence $\{d_i^{'}\}_{i=1}^{Y}$ where

$$d_i' = \begin{cases} d_{t+1}, & \text{if } i = t; \\ d_t, & \text{if } i = t+1; \\ d_i, & \text{otherwise.} \end{cases}$$

Also denote $D'_j = \sum_{i=t+1}^{Y} d'_i$. Then

$$D_i = D'_i \quad when \quad i \neq t \tag{A1}$$

and

$$D_i = D_{i+1} + d_{i+1} \tag{A2}$$

Since $\{d_i\}_{i=1}^{Y}$ yields the minimal total cost, there should be

$$\sum_{i=1}^{Y} \sqrt{d_i(d_i\lambda + D_i)} \leq \sum_{i=1}^{Y} \sqrt{d'_i(d'_i\lambda + D'_i)}$$

By (A1), the above is equivalent to

$$\sqrt{d_t(d_t\lambda + D_t)} + \sqrt{d_{t+1}(d_{t+1}\lambda + D_{t+1})} \leq \sqrt{d'_t(d'_t\lambda + D'_t)} + \sqrt{d'_{t+1}(d'_{t+1}\lambda + D'_{t+1})}$$

which by (A2) can be further reduced to

$$\sqrt{d_t(d_t\lambda + D_{t+1} + d_{t+1})} + \sqrt{d_{t+1}(d_{t+1}\lambda + D_{t+1})} \leq \sqrt{d_{t+1}(d_{t+1}\lambda + D_{t+1} + d_t)} + \sqrt{d_t(d_t\lambda + D_{t+1})}.$$

Let $a = d_t(d_t\lambda + D_{t+1} + d_{t+1}), b = d_{t+1}(d_{t+1}\lambda + D_{t+1}), c = d_{t+1}(d_{t+1}\lambda + D_{t+1} + d_t), d = d_t(d_t\lambda + D_{t+1})$. It is easy to verify that

$$a + b = c + d.$$

Also since $d_t < d_{t+1}$, we have

$$|a - b| = |d_t d_{t+1} - (d_{t+1}^2 - d_t^2)\lambda - (d_{t+1} - d_t)D_{t+1}| \leq |d_t d_{t+1} + (d_{t+1}^2 - d_t^2)\lambda + (d_{t+1} - d_t)D_{t+1}| = |c - d|.$$

We can thus apply Lemma A1 and claim $\sqrt{a} + \sqrt{b} \leq \sqrt{c} + \sqrt{d}$. That is,

$$\sqrt{d_t(d_t\lambda + D_{t+1} + d_{t+1})} + \sqrt{d_{t+1}(d_{t+1}\lambda + D_{t+1})} \geq \sqrt{d_{t+1}(d_{t+1}\lambda + D_{t+1} + d_t)} + \sqrt{d_t(d_t\lambda + D_{t+1})}.$$

or equivalently

$$\sum_{i=1}^{Y} \sqrt{d_i(d_i\lambda + D_i)} \geq \sum_{i=1}^{Y} \sqrt{d'_i(d'_i\lambda + D'_i)}.$$

However, this contradicts to our assumption of the minimality of $\sum_{i=1}^{Y} \sqrt{d_i(d_i\lambda + D_i)}$.

Therefore when $\sum_{i=1}^{Y} \sqrt{d_i(d_i\lambda + D_i)}$ gets the minimum value, there should be $d_1 \geq \cdots \geq d_Y$. $\square$