Supplement to "Repeated Newsvendor Game with Transshipments under Dual Allocations" – Technical Appendix

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Appendix A

Theorem 3. *In an inventory-sharing game with n symmetric retailers facing strictly increasing and independent distribution functions, there is an $M > 0$ such that δ_n^* is decreasing in n for $n \geq \hat{n}$, where $\hat{n} = \min\{n \in Z : nX^d \geq M\}$*[1]*.*

Proof of Theorem 3: In order to prove this theorem, we first introduce the following notation: let $F^m(y) = P\{\sum_{i=1}^m D_i \leq y\}$, $\hat{F}^m(y) = P\{\frac{1}{m}\sum_{i=1}^m D_i \leq y\}$, and $\mathsf{E}[D_i] = \mu$. Note that $F^m(y) = \hat{F}^m(\frac{y}{m})$. We will also need the following lemmas.

Lemma A1. *In an inventory-sharing game with symmetric retailers facing strictly increasing and independent distribution functions, a retailer defecting from strategy $(X^d, \bar{H}_i, \bar{E}_i)$ maximizes her benefit from defection if she orders X^d.*

Proof of Lemma A1: If we have n symmetric retailers, the dual price of retailer i's residual will be either 0 or p, depending on the amount she is sharing with the others. For example, if $\sum_{j\neq i}(\bar{E}_j - \bar{H}_j) = k > 0$, the retailers other than i need k additional units of products. Then, retailer i will receive p per unit if $0 < \bar{H}_i < k$, while she will get nothing otherwise. More formally, retailer i's total expected profit when she orders X_i and other retailers order $\mathbf{X}_{-i}^d$ is given by

$$\begin{aligned} J_i^d(X_i|\mathbf{X}_{-i}^d) \quad &= \quad r\mathsf{E}[\min\{X_i,\, D_i\}] + v\mathsf{E}[H_i] - cX_i + p\int_0^\infty f^{n-1}\Big((n-1)X^d + k\Big)\int_{X_i-k}^{X_i}(X_i - u)f(u)dudk + \\ &\quad p\int_0^\infty f^{n-1}\Big((n-1)X^d - k\Big)\int_{X_i}^{X_i+k}(u - X_i)f(u)dudk, \end{aligned}$$

where $f^{n-1}((n-1)X^d + y)$ is the probability density when the residual demand (resp., inventory)

[1] If D has a finite support with upper bound $\bar{D}$, then $M = \bar{D}$.

for the remaining $(n-1)$ retailers is $y > 0$ (resp., $(-y) > 0$), and its first derivative is given by

$$
\begin{aligned}
(J_i^d)'(X_i|\mathbf{X}_{-i}^d) \quad = \quad & r - c - (r-v)F(X_i) + p\int_0^\infty [F(X_i) - F(X_i - k)] f^{n-1}\left((n-1)X^d + k\right) dk - \\
& p\int_0^\infty [F(X_i + k) - F(X_i)] f^{n-1}\left((n-1)X^d - k\right) dk - \qquad \text{(A1)} \\
& p\int_0^\infty k\left[f(X_i - k) f^{n-1}\left((n-1)X^d + k\right) - f(X_i + k) f^{n-1}\left((n-1)X^d - k\right)\right] dk.
\end{aligned}
$$

Retailer i can increase her profit if she deviates whenever her dual price is zero. In other words, she maximizes her profit if she withholds part of her residual inventory/demand to make it lower than the total residual demand/inventory from other retailers. Under this kind of strategy, her total expected profit will be increased to

$$
\begin{aligned}
J_i^{def}(X_i|\mathbf{X}_{-i}^d) \quad = \quad & r\mathsf{E}[\min\{X_i,\, D_i\}] + v\mathsf{E}[H_i] - cX_i + p\int_0^\infty f^{n-1}\left((n-1)X^d + k\right)\int_{X_i-k}^{X_i}(X_i - u)f(u)du dk + \\
& p\int_0^\infty f^{n-1}\left((n-1)X^d - k\right)\int_{X_i}^{X_i+k}(u - X_i)f(u)du dk + \\
& p\int_0^\infty k f^{n-1}\left((n-1)X^d + k\right) F(X_i - k)dk + p\int_0^\infty k f^{n-1}\left((n-1)X^d - k\right)[1 - F(X_i + k)]dk,
\end{aligned}
$$

and its derivatives are

$$
\begin{aligned}
(J_i^{def})'(X_i|\mathbf{X}_{-i}^d) \quad = \quad & r - c - (r-v)F(X_i) + p\int_0^\infty [F(X_i) - F(X_i - k)]\, f^{n-1}\left((n-1)X^d + k\right) dk - \\
& p\int_0^\infty [F(X_i + k) - F(X_i)]\, f^{n-1}\left((n-1)X^d - k\right) dk, \qquad \text{(A2)} \\
(J_i^{def})''(X_i|\mathbf{X}_{-i}^d) \quad = \quad & -tf(X_i) - p\int_0^\infty \Big[f(X_i - k) f^{n-1}\left((n-1)X^d + k\right) + \\
& f(X_i + k) f^{n-1}\left((n-1)X^d - k\right)\Big] dk < 0.
\end{aligned}
$$

Because all demands follow an identical distribution, it follows from (A1) and (A2) that

$$
\begin{aligned}
[(J_i^{def})' - (J_i^d)'](X_i|\mathbf{X}_{-i}^d) \quad & = \quad p\int_0^\infty k\left[f(X_i - k) f^{n-1}\left((n-1)X^d + k\right) - f(X_i + k) f^{n-1}\left((n-1)X^d - k\right)\right] dk \\
& = \quad E\left[X_i - D_i|\sum_{m=1}^n D_m = (n-1)X^d + X_i\right] = \frac{n-1}{n}\left(X_i - X^d\right).
\end{aligned}
$$

Recall that $X^d = \arg\max J_i^d(X_i|\mathbf{X}_{-i}^d)$, and consequently $(J_i^d)'(X^d|\mathbf{X}_{-i}^d)) = 0$. This implies

$$
(J_i^{def})'(X^d|\mathbf{X}_{-i}^d) = (J_i^d)'(X^d|\mathbf{X}_{-i}^d) + [(J_i^{def})'(X^d|\mathbf{X}_{-i}^d) - (J_i^d)'(X^d|\mathbf{X}_{-i}^d)] = 0 + \frac{n-1}{n}(X^d - X^d) = 0.
$$

Since $J_i^{def}(X_i|\mathbf{X}_{-i}^d)$ is a concave function, the optimal ordering decision when player i defects, X_i^{def}, should satisfy $(J_i^{def})'(X_i^{def}|\mathbf{X}_{-i}^d)) = 0$. Thus, $X_i^{def} = X^d$, and a retailer contemplating a defection maximizes her profit if she orders at the decentralized optimal level. $\square$

Lemma A2. *In an inventory-sharing game with n symmetric retailers and strictly increasing demand distribution function, the expected profit for each retailer, $J^d\left(X^d(n), n\right)$, is increasing in n, where $X^d(n)$ is the NE ordering decision for each retailer in the decentralized system.*

Proof of Lemma A2: Consider a game with $n+1$ symmetric retailers, and let $\mathcal{S}$ be any n-members subset of these retailers. In terms of cooperative game theory, the value of the coalition $\mathcal{S}$ corresponds to the profit generated by its members; because the retailers are symmetric, it can be written as $V^*_{\mathcal{S}} = nJ^d(X,\, n)$, where $J^d(X,\, n)$ denotes the expected profit generated by an arbitrary retailer in a game with n symmetric retailers under dual allocations. However, in an $(n+1)$-retailer game with dual allocations, each retailer will receive a payoff $J^d(X,\, n+1)$. Because dual allocations belong to the core, we must have $nJ^d(X,\, n+1) > V^*_{\mathcal{S}} = nJ^d(X,\, n)$. It is then straightforward that $J^d\left(X^d(n+1), n+1\right) \geq J^d\left(X^d(n), n+1\right) \geq J^d\left(X^d(n), n\right)$. $\square$

We can now prove the theorem. Consider the model with n symmetric retailers and suppose that there were no prior defections. That is, each retailer orders X^d and shares her entire residuals. Recall that we have shown in Lemma A1 that defecting retailers maximize their profit if they order X^d and deviate in the amount they share with others. Under demand realization $\mathbf{D}$, let $\bar{P}_i^{def}(\mathbf{X}^d, \mathbf{D}, n)$ denote the highest payoff that retailer i can generate if she defects in a game with n players, while the other retailers cooperate, and recall that $P_i^d(\mathbf{X}^d, \mathbf{D}, n)$ is her profit in the current period if she shares all of her residuals. After defection, she will receive $J_i(X_1)$ in all subsequent periods. Thus, a possible deviation by player i is deterred if her discount factor satisfies

$$\bar{P}_i^{def}(\mathbf{X}^d, \mathbf{D}, n) + \frac{\delta}{1-\delta} J_i(X_1) < \frac{\delta}{1-\delta} J_i^d(\mathbf{X}^d, n) + P_i^d(\mathbf{X}^d, \mathbf{D}, n), \forall \mathbf{D}, \tag{A3}$$

where $J_i^d(\mathbf{X}^d, n)$ denotes the payoff that retailer i receives when n retailers use dual allocations, order $\mathbf{X}^d$, and share their entire residuals. It is easy to verify that (A3) holds whenever

$$\delta > \delta_{i,n} = \frac{1}{1 + \frac{J_i^d(\mathbf{X}^d, n) - J_i(X_1)}{\sup_{\mathbf{D}}\{\bar{P}_i^{def}(\mathbf{X}^d, \mathbf{D}, n) - P_i^d(\mathbf{X}^d, \mathbf{D}, n)\}}}. \tag{A4}$$

Note that the upper bound of the extra profit that one can get out of deviation, $\sup_{\mathbf{D}}\{\bar{P}_i^{def}(\mathbf{X}^d, \mathbf{D}, n) - P_i^d(\mathbf{X}^d, \mathbf{D}, n)\}$, can be obtained by comparing two cases: (i) the extra profit generated when $D_i = 0$ and the total residual demand of the remaining retailers is slightly below X^d; and (ii) the extra profit generated when $D_{-i} = 0$ and D_i is slightly above nX^d. In the first case, this profit is pX^d; in the second case, this profit would be $p(n-1)X^d$, assuming that demand can achieve values above nX^d. However, note that in most real-life situations there is an $M > 0$ such that $P(D_i > M)$ is negligible (if demand distribution has a finite support with upper bound $\bar{D}$, then $M = \bar{D}$), and the maximum benefit from defection is $p(M - X^d)$. Let us denote $\hat{n} = \min\{n : nX^d \geq M\}$. Then, whenever $n \geq \hat{n}$, it implies that $\sup_{\mathbf{D}}\{\bar{P}_i^{def}(\mathbf{X}^d, \mathbf{D}, n) - P_i^d(\mathbf{X}^d, \mathbf{D}, n)\} = \max\{pX^d, p(M - X^d)\}$, and (A4) corresponds to

$$\delta > \delta_{i,n} = \frac{p \max\{X^d, M - X^d\}}{p \max\{X^d, M - X^d\} + J_i^d(\mathbf{X}^d, n) - J_i(X_1)}.$$

Because the players are symmetric, let $\delta_n = \delta_{i,n}$. Since $J_i(X_1)$ does not depend on n and we showed in Lemma A2 that $J_i^d(\mathbf{X}^d, n)$ increases with n, δ_n is decreasing in n. Finally, let $\delta_n^* = \delta_n$. $\square$

Proposition 2. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, the asymptotic behavior of the equilibrium ordering quantity can be described by*

$$\lim_{n\to\infty} X^d(n) = \begin{cases} \mu, & \text{if } t=0 \text{ or } \frac{r-c-p}{t} \le F(\mu) \le \frac{r-c}{t}, \\ \sup\{x : F(x) < \frac{r-c}{t}\} & \text{if } F(\mu) > \frac{r-c}{t}, \\ \inf\{x : F(x) > \frac{r-c-p}{t}\} & \text{if } F(\mu) < \frac{r-c-p}{t}. \end{cases}$$

Proof of Proposition 2: When each retailer orders X^d, the total expected profit for each of them can be determined by

$$\begin{aligned} J(\mathbf{X}^d) &= r\mathsf{E}[\min\{X^d,\, D\}] + v\mathsf{E}[H] - cX^d + \\ &\quad p\int_0^\infty kf(X^d-k)\left[1-\hat{F}^{n-1}\left(X^d+\frac{k}{n-1}\right)\right]dk + p\int_0^\infty kf(X^d+k)\hat{F}^{n-1}\left(X^d-\frac{k}{n-1}\right)dk \\ &= (r-c)X^d - (r-v)\left[X^dF(X^d) - \int_0^{X^d} yf(y)dy\right] + \\ &\quad p\int_0^\infty kf(X^d-k)\left[1-\hat{F}^{n-1}\left(X^d+\frac{k}{n-1}\right)\right]dk + p\int_0^\infty kf(X^d+k)\hat{F}^{n-1}\left(X^d-\frac{k}{n-1}\right)dk. \end{aligned}$$

If we let $\sigma^2 = Var[D_i]$, then by the Central Limit Theorem (CLT) we have

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m D_i \sim N\left(\mu, \frac{\sigma^2}{m}\right).$$

Suppose first that $X^d > \mu$. Then, we have $\lim_{n\to\infty}[1-\hat{F}^{n-1}(X^d+\frac{k}{n-1})] = 0$ and $\lim_{n\to\infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 1$, hence the derivative of $J(\cdot|\mathbf{X}^d_{-i})$ evaluated at X^d becomes

$$J'(X^d|\mathbf{X}^d_{-i}) \;=\; r-c-(r-v)F(X^d) - p + pF(X^d) = -(c-v) + t[1-F(X^d)],$$

which is a decreasing function of X^d. Thus, if $t=0$ or $F(\mu) \ge 1 - \frac{c-v}{t} = \frac{r-c-p}{t}$, then $J'(X^d|\mathbf{X}^d_{-i}) \le 0$ for any $X^d \in (\mu,\infty)$, and the retailer maximizes her profit by choosing $X^d \to \mu^+$. Otherwise, $X^d = \inf\{x : F(x) > \frac{r-c-p}{t}\}$ is an optimal solution within (μ,∞).

If $X^d < \mu$, $\lim_{n\to\infty}[1-\hat{F}^{n-1}(X^d+\frac{k}{n-1})] = 1$ and $\lim_{n\to\infty}\hat{F}^{n-1}(X^d-\frac{k}{n-1}) = 0$. The derivative of $J(\cdot|\mathbf{X}^d_{-i})$ evaluated at X^d becomes

$$J'(X^d|\mathbf{X}^d_{-i}) \;=\; r-c-(r-v)F(X^d) + pF(X^d) = (r-c) - tF(X^d),$$

which is again a decreasing function of X^d. In this case, if $F(\mu) \le \frac{r-c}{t}$ or $t=0$, then $J'(X^d|\mathbf{X}^d_{-i}) \ge 0$ for any $X^d \in (\infty,\mu)$, and the retailer maximizes her profit by choosing $X^d \to \mu^-$. Otherwise, $X^d = \sup\{x : F(x) < \frac{r-c}{t}\}$ is an optimal solution within $(-\infty,\mu)$.

From the above, we can conclude that whenever $F(\mu) \in [\frac{r-c-p}{t}, \frac{r-c}{t}]$ or $t = 0$, the retailer should select $X^d \to \mu$. Otherwise, because $\frac{r-c-p}{t} \leq \frac{r-c}{t}$, any local optimum is also a global optimum whenever $F(\mu) \notin [\frac{r-c-p}{t}, \frac{r-c}{t}]$. $\square$

Corollary 1. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, the following relationships hold when n is large:*

1. *When $t > 0$: if $F(\mu) > \frac{r-c}{t}$, then $X^1 \leq X^d(n) < \mu$; if $F(\mu) < \frac{r-c-p}{t}$, then $\mu < X^d(n) \leq X^1$.*
2. *When $t = 0$: if $F(\mu) > \frac{r-c}{r-v}$, then $X^1 \leq X^d(n) = \mu$; if $F(\mu) < \frac{r-c}{r-v}$, then $X^1 \geq X^d(n) = \mu$.*

Proof of Corollary 1: Suppose first that $t > 0$. If $F(\mu) > \frac{r-c}{t}$, it follows from Proposition 2 that $\lim_{n\to\infty} X^d(n) = \sup\{x : F(x) < \frac{r-c}{t}\}$. This implies that $F(X^d) \leq \frac{r-c}{t} < F(\mu)$, hence $X^d < \mu$. On the other hand, when there is no cooperation among the retailers, the optimal ordering level X^1 can be determined by the newsvendor model, $F(X^1) = \frac{r-c}{r-v}$. Recall that we assume $p = r - v - t \geq 0$, which implies $r - v \geq t$, therefore $F(X^1) \leq F(X^d)$, and $X^1 \leq X^d$.

If, on the other hand, $F(\mu) < \frac{r-c-p}{t}$, then $\lim_{n\to\infty} X^d(n) = \inf\{x : F(x) > \frac{r-c-p}{t}\}$. This implies that $F(\mu) < \frac{r-c-p}{t} \leq F(X^d)$, hence $\mu < X^d$. Consequently, $F(X^1) = \frac{r-c}{r-v} \geq \frac{r-c-p}{r-v-p} = \frac{r-c-p}{t} = F(X^d)$, so $X^1 \geq X^d$.

When $t = 0$, each retailer orders the expected demand value, and the result is straightforward. $\square$

Theorem 4. *In an inventory-sharing game with n symmetric retailers and strictly increasing distribution function $F(\cdot)$, $\delta_n^* \to \delta_\infty^* > 0$. More specifically, let M be as defined in Theorem 3, and let $\xi(x) = \int_0^x yf(y)dy$ and $\varrho(x) = p\max\{x, M - x\}$. Then,*

$$\delta_\infty^* = \begin{cases} \frac{\varrho(\mu)}{\varrho(\mu)+(r-c-tF(\mu))\mu+t\xi(\mu)-(r-v)\xi(X^1)}, & \text{if } \frac{r-c-p}{t} \leq F(\mu) \leq \frac{r-c}{t} \text{ or } t=0; \\ \frac{\varrho(X^d)}{\varrho(X^d)+t\xi(X^d)-(r-v)\xi(X^1)}, & \text{if } F(\mu) > \frac{r-c}{t} \text{ and } X^d = \sup\{x : F(x) < \frac{r-c}{t}\}; \\ \frac{\varrho(X^d)}{\varrho(X^d)+t(\xi(X^d)-\mu)-(r-v)(\xi(X^1)-\mu)}, & \text{if } F(\mu) < \frac{r-c-p}{t} \text{ and } X^d = \sup\{x : F(x) > \frac{r-c-p}{t}\}. \end{cases}$$

Proof of Theorem 4: Recall that the lower bound of δ_n satisfies

$$\delta_n^* = \frac{p\max\{X^d, M - X^d\}}{p\max\{X^d, M - X^d\} + J_i^d(\mathbf{X}^d, n) - J_i(X_1)} = \frac{\rho(X^d)}{\rho(X^d) + J_i^d(\mathbf{X}^d, n) - J_i(X_1)} \forall i. \tag{A5}$$

In addition, in the model without cooperation, each retailer's profit is maximized at $X^1 = F^{-1}\left(\frac{r-c}{r-v}\right)$, and equals

$$J^1(X^1) = (r - v)\int_0^{X^1} yf(y)dy = (r - v)\xi(X^1). \tag{A6}$$

If $X^d = \mu$, it follows from the CLT that $\lim_{n\to\infty} 1-\hat{F}^{n-1}(X^d+\frac{k}{n-1}) = \lim_{n\to\infty} \hat{F}^{n-1}(X^d-\frac{k}{n-1}) = \frac{1}{2}$, which implies

$$\begin{aligned}
J_i^d(\mathbf{X}^d, n) &= (r-c)X^d - (r-v)\left[X^d F(X^d) - \int_0^{X^d} y f(y)dy\right] \\
&\quad + p\int_0^{\infty} k f(X^d-k)\left[1-\hat{F}^{n-1}\left(X^d+\frac{k}{n-1}\right)\right]dk + p\int_0^{\infty} k f(X^d+k)\hat{F}^{n-1}\left(X^d-\frac{k}{n-1}\right)dk \\
&= (r-c)\mu - (r-v)\left[\mu F(\mu) - \int_0^{\mu} y f(y)dy\right] + \frac{p}{2}\left[\int_0^{\infty} k f(\mu-k)dk + \int_0^{\infty} k f(\mu+k)dk\right] \\
&= [r-c-tF(\mu)]\mu + t\int_0^{\mu} y f(y)dy \\
&= [r-c-tF(\mu)]\mu + t\xi(\mu) \qquad \text{(A7)}
\end{aligned}$$

By substituting (A6) and (A7) into (A5), we obtain

$$\delta_\infty^* = \frac{\rho(\mu)}{\rho(\mu) + [r-c-tF(\mu)]\mu + t\xi(\mu) - (r-v)\xi(X^1)}.$$

If $X^d = \sup\{x : F(x) < \frac{r-c}{t}\} < \mu$, we have $\lim_{n\to\infty} 1-\hat{F}^{n-1}(X^d+\frac{k}{n-1}) = 1$ and $\lim_{n\to\infty} \hat{F}^{n-1}(X^d-\frac{k}{n-1}) = 0$, hence

$$\begin{aligned}
J_i^d(\mathbf{X}^d, n) &= (r-c)X^d - (r-v)\left[X^d F(X^d) - \int_0^{X^d} y f(y)dy\right] \\
&\quad + p\int_0^{\infty} k f(X^d-k)\left[1-\hat{F}^{n-1}\left(X^d+\frac{k}{n-1}\right)\right]dk + p\int_0^{\infty} k f(X^d+k)\hat{F}^{n-1}\left(X^d-\frac{k}{n-1}\right)dk \\
&= (r-c)X^d - (r-v)\left[X^d F(X^d) - \int_0^{X^d} y f(y)dy\right] + p\int_0^{\infty} k f(X^d-k)dk \\
&= t\int_0^{X^d} y f(y)dy \\
&= t\xi(X^d) \qquad \text{(A8)}
\end{aligned}$$

By substituting (A6) and (A8) into (A5), we obtain

$$\delta_\infty^* = \frac{\rho(X^d)}{\rho(X^d) + t\xi(X^d) - (r-v)\xi(X^1)}.$$

Finally, if $X^d = \inf\{x : F(x) > \frac{r-c-p}{t}\} > \mu$, we have $\lim_{n\to\infty} 1-\hat{F}^{n-1}(X^d+\frac{k}{n-1}) = 0$ and

$\lim_{n\to\infty} \hat{F}^{n-1}(X^d - \frac{k}{n-1}) = 1$, hence

$$\begin{aligned}
J_i^d(\mathbf{X}^d, n) &= (r-c)X^d - (r-v)\left[X^d F(X^d) - \int_0^{X^d} y f(y) dy\right] \\
&\quad + p\int_0^{\infty} k f(X^d - k)\left[1 - \hat{F}^{n-1}\left(X^d + \frac{k}{N-1}\right)\right] dk + p\int_0^{\infty} k f(X^d + k)\hat{F}^{n-1}\left(X^d - \frac{k}{n-1}\right) dk \\
&= (r-c)X^d - (r-v)\left[X^d F(X^d) - \int_0^{X^d} y f(y) dy\right] + p\int_0^{\infty} k f(X^d + k) dk \\
&= p\mu + t\int_0^{X^d} y f(y) dy \\
&= p\mu + t\xi(X^d) \qquad \text{(A9)}
\end{aligned}$$

By substituting (A6) and (A9) into (A5), we obtain

$$\delta_\infty^* = \frac{\rho(X^d)}{\rho(X^d) + p\mu + t\xi(X^d) - (r-v)\xi(X^1)}.$$

□

Proposition 5. *If n retailers face i.i.d. demand distributions and differ only in their material costs (that is, $r_i = r_j = r, v_i = v_j = v, t_{ij} = t_{ji} = t$ for $i, j \in \{1, \ldots, n\}$), a first-best outcome cannot be achieved.*

Proof of Proposition 5: Retailers have the same demand distribution $F(\cdot)$, price ,r, salvage value, v, transshipping cost, t, and unit profit from transshipment, $p = r - v - t$. Denote $X = \sum_j X_j$, $X_{-i} = \sum_{j\neq i} X_j$ and let f^m the $p.d.f$ of mD_i. It can be verified that

$$\begin{aligned}
\frac{\partial J_i^d}{\partial X_i} - \frac{\partial J^n}{\partial X_i} &= p\int_0^{\infty} k f(X_i - k) f^{n-1}(X_{-i} + k) dk - p\int_0^{\infty} k f(X_i + k) f^{n-1}(X_{-i} - k) dk \\
&= p\,\mathsf{E}[X_i - D_i \,|\, X = D]\, f^n(X)
\end{aligned}$$

Denote $O_i = \left(\frac{\partial J_i^d}{\partial X_i} - \frac{\partial J^n}{\partial X_i}\right)|_{\mathbf{X}^n}$. Achieving first best requires $O_i = 0$ for all i. However, for any $i \neq j$,

$$\begin{aligned}
O_i - O_j &= p\, f^n(X)\, \mathsf{E}[X_i^n - X_j^n + D_j - D_i \,|\, D = X] \\
&= p\, f^n(X)\, \left[X_i^n - X_j^n + \mathsf{E}[D_j - D_i \,|\, D = X]\right] \\
&= p\, f^n(X)\, (X_i^n - X_j^n)
\end{aligned}$$

It therefore requires $X_i^n = X_j^n$, $\forall i, j$. This is obviously not true given that each X_i^n has to satisfy its FOC with a different c_i:

$$\frac{\partial J^n}{\partial X_i^n} = r - c_i + (r-v)F(X_i^n) + p\, Pr\{D_i \le X_i^n,\, D > X^n\} - p\, Pr\{D_i \ge X_i^n,\, D < X^n\} = 0.$$

□

Theorem 5. *Suppose that all retailers participate in inventory sharing and $J^n(\mathbf{X})$ is unimodal. Then, the eviction contract* $(\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{H}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{E}_t(\hat{\mathbf{h}}_{t-1}), \mathbf{d}_t(\hat{\mathbf{h}}_t), \mathbf{B})$ *is a contract that induces a first-best solution if the retailers' ordering strategies,* $\mathbf{X}_t$, *are given by*

$$\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}|Z_t = Z^k) = \mathbf{X}^k,$$

all coalition members share their entire residuals, the evicted members share nothing, the discretionary transfer payments are

$$d_{it}(\hat{\mathbf{h}}_t) = \begin{cases} \frac{\Delta_{it}(\hat{\mathbf{h}}_t)}{\sum_{I_t^+} \Delta_{jt}(\hat{\mathbf{h}}_t)} \times \sum_{I_t^-} (-\Delta_{jt}(\hat{\mathbf{h}}_t)) & i \in I_t^+ \\ \Delta_{it}(\hat{\mathbf{h}}_t) & i \in I_t^-, \end{cases}$$

where

$$\Delta_{it}(\hat{\mathbf{h}}_t) = \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t),$$
$$I_t^+ = \{i : \Delta_{it}(\hat{\mathbf{h}}_t) > 0\} \quad and \quad I_t^- = \{i : \Delta_{it}(\hat{\mathbf{h}}_t) \leq 0\},$$

and the one-time contract activation bonus is given as

$$B_i = \begin{cases} \frac{\Lambda_i}{\sum_{K^-} \Lambda_i} \times \sum_{K^+} (-\Lambda_i) & i \in K^- \\ \Lambda_i & i \in K^+, \end{cases}$$

where

$$\Lambda_i = \frac{1}{1-\delta_i}(J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)), \quad K^+ = \{i : \Lambda_i > 0\} \quad and \quad K^- = \{i : \Lambda_i \leq 0\}.$$

Proof of Theorem 5: The eviction contract described in Theorem 5 will be an optimal contract if it satisfies the following constraints:

1. *Participation constraint* – each retailer is better off if she adopts the contract;

2. *Early adoption constraint* – each retailer prefers to adopt the contract in the current period than in the later period;

3. *Continuation constraints* – each retailer is better off if she does not deviate in any period.

We now show that the eviction contract satisfies all three constraints.

PARTICIPATION CONSTRAINT: If retailer i adopts the contract in period 1, her infinite horizon discounted payoff is given by

$$B_i + \sum_{t=1}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^n) = B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

If the contract is not adopted and each retailer orders the individually optimal quantity (under the dual allocation rule), her payoff is

$$\sum_{t=1}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^d) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d).$$

The participation constraint is satisfied if

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \geq \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)]$. In other words, retailer i's profit is larger if the retailers order $\mathbf{X}^d$, and she receives a positive bonus to compensate for ordering $\mathbf{X}^n$. Then,

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d),$$

and hence i is not better off if she does not adopt the contract.

Now, suppose that $\Lambda_i \leq 0$ – that is, retailer i's profit is larger if the retailers order $\mathbf{X}^n$ and she gives a side payment to other retailers to induce their acceptance of the contract. Observe that $J^n(\mathbf{X}^n) \geq J^n(\mathbf{X}^d)$, which implies $\sum_i \Lambda_i \leq 0$. This further means that $0 \leq \sum_{K^+} \Lambda_j \leq \sum_{K^-} (-\Lambda_j)$ and

$$0 \leq \frac{\sum_{K^+}(-\Lambda_j)}{\sum_{K^-} \Lambda_j} \leq 1. \tag{A10}$$

Now,

$$\begin{aligned} B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) &= \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] \times \frac{\sum_{K^+}(-\Lambda_j)}{\sum_{K^-} \Lambda_j} + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \\ &\geq \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d), \end{aligned}$$

where the inequality follows from (A10). Thus, the participation constraint is satisfied for all i.

Early adoption constraint: If the contract is adopted in period $t = 2$ instead of in period $t = 1$, the retailers order $\mathbf{X}^d$ in period 1, and retailer i realizes the payoff

$$J_i^n(\mathbf{X}^d) + \delta_i B_i + \sum_{t=2}^{\infty} \delta_i^{t-1} J_i^n(\mathbf{X}^n) = J_i^n(\mathbf{X}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

The early adoption constraint holds if

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \geq J_i^n(\mathbf{X}) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n).$$

First, suppose that $\Lambda_i > 0$, which implies $B_i = \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)]$. Then,

$$J_i^n(\mathbf{X}^d) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) = J_i^n(\mathbf{X}^d) + \frac{\delta_i}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d),$$

and

$$B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) \quad = \quad \frac{1}{1-\delta_i}[J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) = \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^d).$$

Hence, retailer i does not benefit from late adoption of the contract.

Next, when $\Lambda_i \leq 0$, then $J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n) \leq 0$, and (A10) implies

$$\begin{aligned} & B_i + \frac{1}{1-\delta_i} J_i^n(\mathbf{X}^n) - \left(J_i^n(\mathbf{X}^d) + \delta_i B_i + \frac{\delta_i}{1-\delta_i} J_i^n(\mathbf{X}^n) \right) = \\ & = [J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n)] \times \frac{\sum_{K^+}(-\Lambda_j)}{\sum_{K^-}\Lambda_j} + J_i^n(\mathbf{X}^n) - J_i^n(\mathbf{X}^d) \\ & \geq J_i^n(\mathbf{X}^d) - J_i^n(\mathbf{X}^n) + J_i^n(\mathbf{X}^n) - J_i^n(\mathbf{X}^d) = 0. \end{aligned}$$

Thus, retailer i prefers to adopt the contract in the first period.

Continuation constraint: We now want to show that a retailer never benefits from defecting. Recall that Z_t denotes the coalition structure in period t, and suppose that retailer i orders a quantity different from $X_{it}^{Z_t}$ and/or withhold some of her residuals. As a result, she pays a penalty, d_{it}, in period t, and is excluded from inventory sharing in all subsequent periods. We denote, with slight abuse of notation, $\mathbf{X}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{X}_t$, $\mathbf{H}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{H}_t$, $\mathbf{E}_t(\hat{\mathbf{h}}_{t-1}) = \mathbf{E}_t$, $d_{it}(\hat{\mathbf{h}}_t) = d_{it}$, and $\Delta_{it}(\hat{\mathbf{h}}_t) = \Delta_{it}$. Then, retailer i's discounted payoff starting from period t is given by

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + d_{it}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1).$$

The continuation constraint holds if

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + d_{it}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) \leq \frac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}).$$

If $i \in I_t^-$, then $\Delta_{it} \leq 0$, and $d_{it} = \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t)$. Thus, i receives a payoff

$$J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \frac{\delta_i}{1-\delta_i} J_i^1(X_i^1) = \frac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}),$$

and i does not benefit from defection.

Now, suppose $i \in I_t^+$, and consequently $\Delta_{it} > 0$. This implies

$$\frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \geq 0. \tag{A11}$$

Notice that

$$\begin{aligned} \sum_i \Delta_{it} \quad &= \quad \sum_i \left\{ \frac{1}{1-\delta_i}\left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1)\right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \right\} \\ &= \quad \frac{\delta_i}{1-\delta_i}\left\{J^{Z_t}(\mathbf{X}^{Z_t}) - J^1(\mathbf{X}_1)\right\} + J^{Z_t}(\mathbf{X}^{Z_t}) - J^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \geq 0 \end{aligned}$$

where the inequality holds because $\mathbf{X}^{Z_t}$ with complete residual sharing maximizes the system profit when the state is Z_t and systems with inventory-sharing retailers generate higher profit than systems without inventory sharing. As a result, $\sum_{I_t^+} \Delta_{jt} \geq \sum_{I_t^-} (-\Delta_{jt})$, and

$$0 \leq \frac{\sum_{I_t^-} (-\Delta_{jt})}{\sum_{I_t^+} \Delta_{jt}} \leq 1. \tag{A12}$$

Thus, retailer i receives a payoff

$$\begin{aligned} &J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \left\{ \tfrac{1}{1-\delta_i} \left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) \right\} \times \tfrac{\sum_{I_t^-} (-\Delta_{jt})}{\sum_{I_t^+} \Delta_{jt}} + \tfrac{\delta_i}{1-\delta_i} J_i^1(X_i^1) \\ &\quad \leq J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \tfrac{1}{1-\delta_i} \left[J_i^{Z_t}(\mathbf{X}^{Z_t}) - \delta_i J_i^1(X_i^1) \right] - J_i^{Z_t}(\mathbf{X}_t, \mathbf{H}_t, \mathbf{E}_t) + \tfrac{\delta_i}{1-\delta} J_i^1(X_i^1) = \tfrac{1}{1-\delta_i} J_i^{Z_t}(\mathbf{X}^{Z_t}), \end{aligned}$$

where the inequality follows from (A11) and (A12). As a result, i prefers not to defect in any period. $\square$

Appendix B - Conditions for achieving a first-best outcome in a repeated game

In this Appendix, we provide some additional interpretation of condition (2). Suppose that retailer i orders quantity X_i. After demand D_i is realized, she may be left with residual demand, $\bar{E}_i$, or with residual supply, $\bar{H}_i$. Suppose that all retailers share their entire residuals, $H_j = \bar{H}_j, E_j = \bar{E}_j, \forall j$. Given $\mathbf{H}_{-i} = (H_1, H_2, \ldots, H_{i-1}, H_{i+1}, \ldots, H_n)$ and $\mathbf{E}_{-i} = (E_1, E_2, \ldots, E_{i-1}, E_{i+1}, \ldots, E_n)$, dual prices λ_j and μ_j will have different values for different E_i and H_i. Note, however, that the values of dual prices change in the form of a step function. λ_j, the price for residual inventory, is non-decreasing with E_i for $j \neq i$. Similarly, μ_j is non-decreasing with H_i. Thus, there are a finite number of jumps for both λ_j and μ_j. In other words, we can find values

$$e_m \in (-X_i, \infty), e_m < e_{m+1}, m = 1, 2, \ldots \quad \text{and} \quad h_l \in (-\infty, X_i), h_l < h_{l+1}, l = 1, 2, \ldots$$

such that given $\mathbf{H}_{-i}$ and $\mathbf{E}_{-i}$, λ_j does not change for any $E_i \in (e_m, e_{m+1})$ (that is, when $D_i \in (X_i + e_m, X_i + e_{m+1})$), and μ_j does not change for any $H_i \in (h_l, h_{l+1})$ (that is, when $D_i \in (X_i - h_l, X_i - h_{l+1})$). At the same time,

$$\begin{aligned} \lambda_j(E_i \in (e_l, e_{l+1}), \mathbf{H}_{-i}, \mathbf{E}_{-i}) &\neq \lambda_j(E_i \in (e_{l+1}, e_{l+2}), \mathbf{H}_{-i}, \mathbf{E}_{-i}) \quad \text{and} \\ \mu_j(H_i \in (h_l, h_{l+1}), \mathbf{H}_{-i}, \mathbf{E}_{-i}) &\neq \mu_j(H_i \in (h_{l+1}, h_{l+2}), \mathbf{H}_{-i}, \mathbf{E}_{-i}). \end{aligned}$$

Thus, we can define

$$\lambda_j^m(\mathbf{H}_{-i}, \mathbf{E}_{-i}) = \lambda_j(E_i \in (e_m, e_{m+1}), \mathbf{H}_{-i}, \mathbf{E}_{-i}), \qquad \mu_j^l(\mathbf{H}_{-i}, \mathbf{E}_{-i}) = \mu_j(H_i \in (h_l, h_{l+1}), \mathbf{H}_{-i}, \mathbf{E}_{-i}).$$

With some abuse of notation, we write λ_j^m and μ_j^l when it is clear what values of $(\mathbf{H}_{-i}, \mathbf{E}_{-i})$ they refer to. We can now define the *total variation* of dual allocations for dual prices λ_j, μ_j w.r.t. retailer i's ordering quantity X_i given the *ex post* residuals of retailers other than i:

$$\begin{aligned}
TV_{\lambda_j}^{(i)}(X_i, \mathbf{H}_{-i}, \mathbf{E}_{-i}) &= H_j \sum_m f_i(X_i + e_m) \left[\lambda_j^{m+1}(\mathbf{H}_{-i}, \mathbf{E}_{-i}) - \lambda_j^m(\mathbf{H}_{-i}, \mathbf{E}_{-i})\right] \\
TV_{\mu_j}^{(i)}(X_i, \mathbf{H}_{-i}, \mathbf{E}_{-i}) &= E_j \sum_l f_i(X_i - h_l) \left[\mu_j^{l+1}(\mathbf{H}_{-i}, \mathbf{E}_{-i}) - \mu_j^l(\mathbf{H}_{-i}, \mathbf{E}_{-i})\right].
\end{aligned}$$

The total variation of dual allocations for dual prices λ_j and μ_j is, therefore, the expected "jump amount" of the allocations resulting from residual supply and residual demand, respectively. We now take the expectation over $\mathbf{H}_{-i}, \mathbf{E}_{-i}$ at $\mathbf{X}_{-i} = \mathbf{X}_{-i}^n$ and denote

$$ETV_{\lambda_j}^{(i)}(X_i) = \mathsf{E}[TV_{\lambda_j}^{(i)}(X_i, \mathbf{H}_{-i}, \mathbf{E}_{-i})],$$

$$ETV_{\mu_j}^{(i)}(X_i) = \mathsf{E}[TV_{\mu_j}^{(i)}(X_i, \mathbf{H}_{-i}, \mathbf{E}_{-i})].$$

We use these expressions to describe the retailers who are "alike" in a more general sense, as follows. In an inventory-sharing game with n retailers, the retailers are *relaxed-symmetric* if, for any i, the sums of the expected total variation w.r.t. i for all retailers other than i are equal for both dual prices if i orders a system-optimal stocking quantity:

$$\sum_{j \neq i} ETV_{\lambda_j}^{(i)}(X_i^n) = \sum_{j \neq i} ETV_{\mu_j}^{(i)}(X_i^n) \qquad \forall i, j. \tag{B1}$$

Observe that, when $n = 2$, (B1) corresponds to the sufficient and necessary condition for achieving a first-best solution (see N&S 2008). This relationship continues to hold when we have an arbitrary number of retailers: our next result follows from Theorem B1 after observing that (B1) corresponds to (2).

Theorem B1. *In an inventory-sharing game with n relaxed-symmetric retailers, if $J^n(\mathbf{X})$ is unimodal in $\mathbf{X}$, a first-best solution can be induced with dual allocations when $\delta > \delta_n^*$.*